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ANALYSIS OF MATHEMATICAL MODELS OF RANDOM GRAPHS

Abstract. Relevance. In the modern world, the majority of social, biological, technological, and communication processes occur in the form of complex network structures. The analysis of such systems requires the construction of mathematical models capable of adequately describing their topology, stochastic nature, and dynamic properties. One of the most powerful tools for this is the theory of random graphs, which enables the modeling of a wide range of real-world phenomena – from the spread of viruses and information to the functioning of critical infrastructures. Classical models, particularly the Erdős-Rényi model, laid the foundation of modern graph theory; however, they have limitations in describing networks with high clustering or uneven degree distributions. Therefore, in recent years, modern approaches have gained special significance, including small-world models, which reflect the properties of real social and biological systems with short paths and a high level of clustering, and scale-free graphs, which model networks with heterogeneous degree distributions. The relevance of this topic is due to the need to select and analyze an appropriate mathematical model that can accurately represent the properties of real networks, enable justified prediction of their behavior, identify vulnerabilities, and optimize the functioning of complex systems. In this context, the analysis of mathematical models of random graphs is an important area of modern applied mathematics, computer science, and systems theory. **The object of research**: Random graphs as mathematical structures that model the topology and dynamics of complex networked systems. **Purpose of the article**: Research, systematization, and comparative analysis of mathematical models of random graphs to determine their suitability for modeling various types of complex networked systems. **Research results**. The article presents an analysis of mathematical models of random graphs that underpin modern network science. Beginning with the theoretical foundations of graphs, both classical and contemporary models for constructing random networks are examined, alongside methods of their analysis and practical areas of application. The methods of analysis are critically important for the practical use of random graphs, as they enable the models to be applied to forecasting, diagnostics, and the management of complex systems. These methods also provide a foundation for integrating mathematical models with real-world data, which is a key objective of contemporary network science. Classical models allow for the formalization of randomness in network connections; however, they have significant limitations in capturing the real topological properties of complex networks – particularly high clustering, degree heterogeneity, and growth mechanisms. In contrast, modern models correspond much more closely to the structure of real-world systems. They make it possible to model such essential features as the emergence of hubs, clustering, short average path lengths, and robustness to failures. **Conclusions.** Mathematical models of random graphs are a universal tool for the analysis and synthesis of various networked systems. Their application allows not only for the formal description of a complex system’s structure, but also for the discovery of its hidden patterns, the prediction of its behavior under different conditions, and the optimization of its functioning while accounting for real-world constraints.

Keywords: random graphs, mathematical modeling, scale-free networks, Erdős-Rényi model, clustering, spectral analysis, network structures.

Introduction

In recent decades, interest in the study of complex networks has grown rapidly, driven by their wide-ranging applications across various fields of science and technology. Social networks, the Internet, biological and ecological systems, as well as transportation and energy infrastructures all share a common feature: they can be described in the form of graphs, where vertices represent system elements and edges represent interactions between them. However, in real-world conditions, such structures are rarely regular or deterministic – they are, as a rule, random. For this reason, mathematical models of random graphs have become a vital tool for the analysis and description of such systems.

Beginning with the classical works of Erdős and Rényi, in which the concept of a random graph was first introduced, the theory has progressed toward more sophisticated models that more accurately describe the properties of real networks. Small-world models have proven effective for representing social and biological networks, where high clustering coexists with short average path lengths. Another important direction involves scale-free models, which reproduce the power-law degree distribution observed in many systems.

Contemporary research focuses on comparing different models, identifying their structural characteristics, adapting them to specific applied problems, and analyzing their computational properties. In this context, a systematic review of existing approaches to modeling random graphs is of relevance – including an assessment of their strengths and limitations, and the formulation of recommendations for model selection depending on the type of network.

Analysis of publications. The analysis of mathematical models of random graphs is a key area of modern network theory, with wide applications in computer science, biology, sociology, and other fields. Among the main models that describe the structure and dynamics of real networks are the classical Erdős-Rényi model, “small-world” models, and scale-free models [1].

The Erdős-Rényi model is one of the first and simplest models of random graphs, in which each pair of vertices relates to a certain probability. This model allows for the study of fundamental graph properties such as the connectivity threshold and the degree distribution of vertices. The “small world” model, proposed by Watts and Strogatz, is characterized by a high clustering coefficient and a small average path length between vertices. This model reflects the properties of many real-world networks, such as social or biological networks, where there is a tendency to form clusters while simultaneously maintaining short paths between any two vertices.

Scale-free models, particularly the Barabási–Albert model, describe networks in which the degree distribution follows a power-law. This means that the network contains a small number of nodes with very high degree, while many nodes have a low degree. Such networks exhibit high resilience to random failures but are vulnerable to targeted attacks on hubs.

Recent studies also explore models that combine properties of small-world and scale-free networks, as well as models that incorporate the geometric and topological features of real systems. These approaches allow for more accurate modeling and analysis of complex network structures found in nature and technology.

In article [2], the emergence of clustering in random graphs with scale-free structure is examined, particularly in the context of hyperbolic models. The authors propose a method for analyzing the probability of triangle formation in graphs that takes vertex degrees into account. The study also considers a model in which vertices are embedded in hyperbolic space, enabling the reproduction of real-world network properties such as clustering and scale-freeness. It is shown that, unlike in some other models, clustering in hyperbolic graphs does not vanish as the network grows. This research represents an important step toward understanding the structure and properties of complex networks, especially in the context of modeling real systems such as social networks or the internet.

The purpose of this work is to analyze the main mathematical models of random graphs, to classify and characterize them, and to evaluate their suitability for use in various contexts, including the modeling of social, technological, and biological networks.

Main part

Mathematical graph theory serves as a foundation for modeling complex systems in which objects interact as nodes connected by links. In random graphs, these links are established not deterministically but with a certain probability, allowing for the modeling of real-world phenomena characterized by uncertainty, variability, and dynamic behavior. Random graphs (Fig. 1) are defined by several important parameters that determine their structure: vertex degree, connected components, graph diameter, average path length, and clustering coefficient. These characteristics make it possible to evaluate both local and global properties of the network.

Classical models of random graphs have formed the theoretical foundation upon which modern network science is built. They allow for the formal description and analysis of the network formation process, the identification of critical thresholds for structural transitions, and the estimation of the probability of the emergence of connected components. Although these models are simplified and often fail to capture the complex nature of real-world networks, they remain an essential step in understanding the basic mechanisms behind link formation between elements.

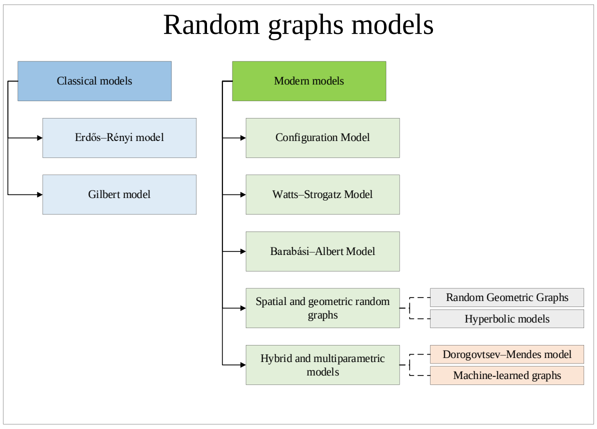

Рис. 1. Random graphs models

The Erdős-Rényi model was the first formalized model of a random graph. This approach assumes that each possible edge is independently included with equal probability pp, generating a set of possible graphs with varying numbers of edges. There is also a version G(n, m), in which exactly m random edges are formed. Graphs produced by this model exhibit a low clustering coefficient and a narrow degree distribution close to binomial. This model is well suited for scenarios where all connections are independent and equally probable, but it does not capture the high clustering or heterogeneity typical of real networks.

The Gilbert model [3], proposed around the same time as the Erdős-Rényi model, is formally very similar: it also defines the probability of edge formation between vertex pairs, but does so in a more general probabilistic framework, without fixing the number of vertices or edges. The Gilbert model is sometimes used as a basis for constructing probabilistic graphs with uneven weight distributions or vertex dependencies, although in practice, the Erdős-Rényi formalism is more commonly employed.

Classical models make it possible to theoretically compute the likelihood of certain structural properties. However, classical graphs exhibit almost no local clustering, which limits their applicability to the modeling of social or biological networks.

Despite their importance, classical models have significant limitations: the degree distribution is symmetric and narrow, which does not correspond to the power-law distributions observed in real networks; they lack mechanisms for growth and preferential attachment; and they feature low clustering levels. These shortcomings have driven the development of modern random graph models that can reflect the heterogeneous nature of connections, clustering, the emergence of hubs, and other non-trivial properties.

With the development of network research, it became evident that classical models of random graphs – particularly the Erdős-Rényi model – are unable to capture many important characteristics of real systems, such as high clustering, heterogeneous degree distributions, or the presence of hubs. To overcome these limitations, modern models have been developed that account for more complex mechanisms of link formation in networks.

The configuration model [4] allows the generation of random graphs with a predefined degree distribution. The main idea is to assign each vertex a certain number of "half-edges" in advance and then randomly pair them to form complete edges.

Key features of this model include flexibility (it can model arbitrary degree distributions, including power-law), low clustering, and frequent use as a null model for real-world networks with specified degree characteristics.

There are also spatial and geometric random graph models. In such models, vertices are embedded in Euclidean or hyperbolic space, and connections between them are determined by distance. For instance, in hyperbolic graphs, distance is defined using non-Euclidean geometry, enabling accurate replication of real network properties. Examples include graphs in which vertices are randomly distributed on a plane and links are formed if the distance between them is below a certain threshold. Hyperbolic models support the modeling of networks that simultaneously exhibit high clustering, power-law degree distributions, and short average path lengths. These models are used in the modeling of internet networks, infrastructure, transport systems, and for optimizing the physical placement of nodes in distributed systems.

Recent studies have introduced hybrid models that combine characteristics of "small-world," scale-free, and spatial graphs. These approaches enable more precise modeling of the specific features of real-world networks. Among them are the Dorogovtsev-Mendes model [5], based on network growth with cyclic attachments, as well as machine learning-based models [6], which employ algorithms to simulate the evolution of network technology.

Modern random graph models demonstrate greater flexibility and alignment with real structures compared to classical ones. They consider not only the probability of connections, but also the mechanisms of their formation, spatial constraints, network growth history, and link heterogeneity.

Analytical methods for studying the properties of random graphs make it possible to determine critical graph parameters, investigate structural dynamics, identify key nodes, assess clustering, and predict network resilience to failures. These methods include probabilistic analysis, spectral analysis, algorithmic and numerical methods, as well as visualization and interpretation techniques.

The probabilistic approach is fundamental in the study of random graphs, as the main models are based on probability. This type of analysis includes: calculating the expected number of connected components, triangles, and degree distributions; estimating the probabilities of the emergence of specific graph properties, such as connectivity, the existence of a giant component, or cycles of a given length; and analyzing phase transitions, where a small change in the parameter pp in the G(n, p) model leads to a sudden structural shift in the graph. This approach utilizes the law of large numbers, Chebyshev's theorems, probabilistic limits, and asymptotic estimates, especially for large n.

Spectral graph theory investigates the properties of the adjacency matrix or the graph Laplacian, which enables the assessment of graph connectivity; the identification of central nodes and groups of nodes with similar spectral properties; and the detection of cluster structures via spectral partitioning. These methods are effective in revealing hidden structures, especially in large and complex networks. Spectral analysis is often used for visualization, clustering, and evaluating the dynamics of spreading processes.

In many cases, analytical investigation of a graph becomes impractical due to its complexity or dimensionality, prompting the use of numerical approaches: simulation modeling (generating a large number of graphs under a given model for statistical analysis); Monte Carlo methods, used to estimate the probabilities of complex events, such as resilience to random or targeted node removals; graph algorithms, applied to identify connected components, calculate centralities, construct minimum spanning trees, etc.; and clustering analysis. Graph visualization tools enable intuitive understanding of network structure, identification of hubs, isolated components, communities, and the analysis of structural changes over time.

Mathematical models of random graphs are widely used in numerous fields where systems can be represented as a set of interconnected objects. With the growth of information technology, bioinformatics, sociology, and cybersecurity, the importance of such models continues to rise. A properly selected model allows for an accurate reproduction of real network structures, analysis of their properties, detection of vulnerabilities, and prediction of system behavior under various scenarios. In sociology, random graphs are used to model interpersonal interactions, contacts, and information exchange. Social networks exhibit "small-world" properties – high clustering and short paths between any two individuals. These models can be applied to study the influence of key individuals, the spread of opinions, rumors, and information, the identification of communities or social groups, and the prediction of behavior in dynamic networks. Models such as those of Watts-Strogatz and Barabási-Albert allow for the reproduction of such network properties, including the presence of hubs (opinion leaders) and cluster structures.

The internet, telephone networks, and peer-to-peer systems are classical examples of communication networks, where nodes represent devices and edges represent communication channels. It has been observed that the internet exhibits a scale-free structure, meaning a small number of nodes have an exceptionally high number of connections. Random graph models can be applied to analyze network resilience to failures, optimize traffic routing, identify vulnerable regions in the topology, and design fault-tolerant architectures.

Biological systems – such as brain neural networks, metabolic pathways, and protein or gene interactions – also exhibit complex network structures. These networks often feature small average path lengths, high clustering, and power-law degree distributions. Graph models can be used to identify critical genes or proteins, analyze signal transmission in neural circuits, simulate failures or mutations, and study functional modules within cells. They enable the investigation of not only topology but also functional interactions among biomolecules within a systems biology context.

One of the most prominent application areas of random graph models is the simulation of diffusion processes, such as the spread of diseases, rumors, or fake information, viral content propagation, and malware dissemination. SIR, SIS, and SEIR models [7] are commonly combined with graph topology to simulate epidemic outbreaks. Understanding topological properties – such as average path length or clustering coefficient – helps predict the scale of spreading and assess the effectiveness of countermeasures.

Power grids, water supply systems, transportation networks, and other critical infrastructures can be modeled as random or deterministic graphs. Key challenges include assessing resilience, optimizing design, and ensuring efficient maintenance. These models are used to identify critical nodes, simulate failure scenarios, optimize resource placement, and design backup communication routes. In complex systems operating in real time, hybrid and spatial graph models are especially valuable, as they account for geographic location, bandwidth constraints, and dynamic load variations. Dorogovtsev-Mendes models introduce recursive growth with feedback, useful for hierarchical systems.

In the process of modeling complex networks, the choice of an appropriate mathematical model is of critical importance. Each model has its own advantages, limitations, structural properties, and domain of effective application. Table 1 presents the results of a comparative analysis of key random graph models based on essential criteria.

Table 1 – **Comparative Analysis of Models**

Model	Degree Distribution	Clustering	Average Path Length
Erdős-Rényi	Binomial	Low	Short
“small-world”	Approximately regular	High	Short
Barabási-Albert	Power-law	Low–medium	Short
Configuration model	User-defined	Low	Depends on input
Hyperbolic random graphs	Power-law	High	Very short
Dorogovtsev-Mendes	Power-law	Medium–high	Short

The analysis has shown that there is no universal model suitable for all types of networks. The selection of a model should be based on the core characteristics of the system under study – such as the nature of the degree distribution, the presence of a clustered structure, the dynamic nature of connections, and spatial or resource constraints. In complex applied problems, the best results are often achieved by hybrid or adaptive models that combine elements of several approaches.

Conclusions

An analysis of mathematical models of random graphs – foundational to modern network science – has been conducted. Starting from the theoretical foundations of graph theory, both classical and contemporary models for constructing random networks have been examined, along with analytical methods and practical applications. Analytical techniques are critically important for the practical use of random graphs, as they make models suitable for prediction, diagnostics, and the management of complex systems. These methods also provide the basis for integrating mathematical models with real-world data, which is a central goal of contemporary network science.

Classical models make it possible to formalize the randomness of connections in a network, but they have significant limitations when it comes to reproducing the real topological features of complex networks – particularly high clustering, degree heterogeneity, and growth mechanisms. Modern models correspond much more closely to the structure of real systems. They allow for the modeling of important features such as the emergence of hubs, clustering, short average path lengths, and robustness to failures.

The Erdős-Rényi model, despite its simplicity and mathematical convenience, has low clustering and does not reflect the uneven structure of connections. Therefore, it is better suited for theoretical studies and as a baseline reference. The Watts-Strogatz model effectively simulates networks with a high level of local cohesion and short average path lengths, which are typical of social and certain biological systems. However, its degree distribution is too uniform to reflect many real-world structures.

The Barabási-Albert scale-free model reproduces the empirical power-law distribution of connections and the emergence of hubs, making it optimal for modeling internet networks, bioinformatics systems, and information node interactions. However, its lack of sufficient clustering limits its application in social contexts unless modified. The configuration model offers high flexibility by allowing arbitrary degree distributions but does not model network growth and requires external information about topology. Spatial and hyperbolic graphs are best suited for modeling infrastructure and transportation networks, where physical node placement and distance constraints play a critical role.

Mathematical models of random graphs serve not only as tools for the formal description of network structures but also as the basis for decision-making, system dynamics prediction, vulnerability detection, and optimization in complex environments.

Ultimately, mathematical models of random graphs are a universal tool for the analysis and synthesis of diverse networked systems. Their application allows not only for the formal description of a complex system’s structure, but also for the discovery of hidden patterns, prediction of behavior under different conditions, and optimization of functionality under real-world constraints.

References

1. Freeze A., Karonski M. Introduction to random graphs. Carnegie Mellon University, 2025. 668 p.
2. Clara Stegehuis, Remco van der Hofstad, Johan S.H. van Leeuwaarden. Scale-free network clustering in hyperbolic and other random. Department of Mathematics and Computer Science. Eindhoven University of Technology, 2018. P.1-19. https://doi.org/10.48550/arXiv.1812.03002
3. Freeze A., Karonski M. Random Graphs and Networks: A First Course. Cambridge University Press, 2023. 232 p. https://doi.org/10.1017/9781009260268
4. Bailey K. Fosdick, Daniel B. Larremore, Joel Nishimura, Johan Ugander. Configuring Random Graph Models with Fixed Degree Sequences. Vol. 60, iss., 2018. P. 315-355. https://doi.org/10.1137/16M1087175
5. S.N. Dorogovtsev, J.F.F. Mendes. Evolution of Networks: From Biological Nets to the Internet and WWW, 1st ed, Oxford University Press,2013. 280 p. https://doi.org/10.1093/acprof:oso/9780198515906.001.0001
6. Wei Ju, Zheng Fang, Yiyang Gu, Zequn Liu, Qingqing Long, Ziyue Qiao, Yifang Qin, Jianhao Shen, Fang Sun, Zhiping Xiao, Junwei Yang, Jingyang Yuan, Yusheng Zhao, Yifan Wang, Xiao Luo, Ming Zhang. A Comprehensive Survey on Deep Graph Representation Learning. Cornell University, 2023. 100 p. https://doi.org/10.48550/arXiv.2304.05055
7. Michael Y. Li. An Introduction to Mathematical Modeling of Infectious Diseases. 1st ed, Springer, 2018. 168 p. https://doi.org/10.1007/978-3-319-72122-4

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Аналіз математичних моделей випадкових графів

В. О. Горбачов, В. М. Федоров

Анотація. Актуальність. У сучасному світі більшість соціальних, біологічних, технологічних і комунікаційних процесів відбувається у вигляді складних мережевих структур. Аналіз таких систем вимагає побудови математичних моделей, здатних адекватно описувати їхню топологію, випадкову природу та динамічні властивості. Одним із найпотужніших інструментів для цього є теорія випадкових графів, що дозволяє моделювати широке коло реальних явищ – від поширення вірусів і інформації до функціонування критичних інфраструктур. Класичні моделі, зокрема модель Ердеша-Реньї, заклали фундамент сучасної теорії графів, однак вони мають обмеження в описі мереж з високою кластеризацією чи нерівномірним розподілом зв'язків. Тому останнім часом особливого значення набули сучасні підходи, серед яких моделі типу «світ тісний», що відображають властивості реальних соціальних чи біологічних систем із короткими шляхами та високим рівнем кластеризації) та безмасштабні графи, що моделюють мережі з нерівномірним розподілом ступенів. Актуальність теми зумовлена необхідністю вибору та аналізу відповідної математичної моделі, яка забезпечить точне представлення властивостей реальних мереж, дозволить проводити обґрунтоване прогнозування їх поведінки, виявляти вразливості та оптимізувати функціонування складних систем. У цьому контексті аналіз математичних моделей випадкових графів є важливим напрямом сучасної прикладної математики, інформатики та теорії систем. **Об'єкт дослідження**: випадкові графи як математичні структури, що моделюють топологію та динаміку складних мережевих систем. **Мета статті**: дослідження, систематизація та порівняльний аналіз математичних моделей випадкових графів для визначення їх придатності до моделювання різних типів складних мережевих систем. **Результати дослідження**. У статті проведено аналіз математичних моделей випадкових графів, що лежать в основі сучасної мережної науки. Починаючи з теоретичних основ графів, було досліджено як класичні, так і сучасні моделі побудови випадкових мереж, методи їх аналізу та практичні напрями застосування. Методи аналізу є критично важливими для практичного використання випадкових графів, оскільки саме вони дозволяють зробити моделі придатними для прогнозування, діагностики та управління складними системами. Вони також формують основу для інтеграції математичних моделей з реальними даними, що є ключовою метою сучасної науки про мережі. Класичні моделі дозволяють формалізувати випадковість зв'язків у мережі, проте мають суттєві обмеження у відтворенні реальних топологічних властивостей складних мереж, зокрема високої кластеризації, неоднорідності ступенів та механізмів зростання. Сучасні моделі значно краще відповідають структурі реальних систем. Вони дозволяють моделювати такі важливі характеристики, як поява хабів, кластеризація, коротка середня довжина шляху та стійкість до збоїв. **Висновки.** Математичні моделі випадкових графів є універсальним інструментом для аналізу та синтезу різноманітних мережевих систем. Їх застосування дозволяє не лише формально описати структуру складної системи, а й виявити її приховані закономірності, передбачити її поведінку за різних умов, а також оптимізувати функціонування з урахуванням реальних обмежень.

Ключові слова: випадкові графи, математичне моделювання, безмасштабні мережі, модель Ердеша–Реньї, кластеризація, спектральний аналіз, мережеві структури.